Regular Implementability and Stabilization Using Controllers With Pre-Specified Input/Output Partition

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Regular Implementability and Stabilization Using Controllers With Pre-Specified Input/Output Partition

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***Abstract*—This paper deals with the problems of regular implementability and stabilization of a given plant in the context of finite-dimensional linear differential system behaviors. In particular we solve the problems of regular implementability and stabilization using controllers in which a pre-specified subset of the plant control variables is free. We will also extend the results to the situation in which the set of plant control variables is partitioned into two complementary subsets. Variables from one subset should become controller inputs, while variables from the other should become controller outputs. In other words, we consider the problems of regular implementability and stabilization using controllers with a priori given input/output structure.**

***Index Terms*—Behaviors, input/outout structure, interconnection, linear systems, regular implementability, stabilization.**

I. Introduction

An important issue in the behavioral approach to control is implementability. Implementability deals with the question: which system behaviors can be achieved by interconnecting a given plant with a controller? In the behavioral framework this is made precise as follows. Given is a plant behavior with two types of variables, the variable w to be controlled and the control variable c. On the control variable c we are allowed to put restrictions. In the behavioral approach we treat a controller as an additional system behavior, called controller behavior. Interconnecting the plant and the controller means that the control variable in the plant should also become an element of the controller behavior. The space of all trajectories w possible after interconnecting the plant with the controller is called the manifest controlled behavior. A behavior is called implementable if it can be obtained as manifest controlled behavior in this way.

In the context of pole placement and stabilization an important role is played by regular implementation. A given behavior is called regularly implementable if it can be achieved by a controller behavior that only imposes restrictions on the control variables different from those of the full plant behavior, equivalently, the output cardinality of the associated full controlled behavior is equal to the sum of the output cardinalities of the plant and the controller. In [12], for a given plant behavior a characterization was given of all implementable behaviors and in [1] a characterization was given of all regularly implementable behaviors.

In many cases, certain components of the plant control variables represent plant sensor measurements, or unknown disturbance inputs to the plant. Such plant variables should obviously not be restricted by the controller, and should therefore be part of the input variables of the controller. In the behavioral framework this is formalized by requiring these plant control variables to be *free* in the controllers that are allowed.

In this paper we deal with the problems of finding necessary and sufficient conditions for a behavior to be regularly implementable using a controller in which an a priori given subset of the plant control variables is free or maximally free, respectively. In other words, we require a priori given components of the plant control variable to be part of controller input, or even to be controller input. The complementary subset in the set of all control variables then necessarily contains the controller output, or is equal to the controller output. This problem was introduced by A. Julius in [3], see also [5]. In the work of Julius, only *sufficient* conditions were obtained, and these conditions were formulated in terms of particular *representations* of the plant and the desired behavior. In the present paper we give necessary and sufficient conditions in terms of the plant behavior and desired behavior. We also introduce the related problem of stabilization by means of controllers in which an a priori given subset of the control variables is free or maximally free. We derive necessary and sufficient conditions for a system to be stabilizable using this kind of controllers. We resolve all these problems for the full as well as for the partial interconnection case, see also [1].

The outline of this paper is as follows. In Sections II and III we review the basic facts on linear systems in the behavioral framework and on the notions of regular implementability and stabilization. In Section IV we formulate the problems of regular implementability and stabilization using controllers in which a priori given control variables should be free or even maximally free. In Sections V and VI we resolve these problems for the full and the partial interconnection case. In Section VII we provide some examples to illustrate the theory presented in this paper. The paper ends with some conclusions in Section VIII.

II. Linear Differential Systems

In the behavioral approach to linear systems, a dynamical system is given by a triple $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{q}}, \mathfrak{B})$, where $\mathbb{R}$ is the time axis, $\mathbb{R}^{\mathtt{q}}$ is the signal space, and the behavior $\mathfrak{B}$ is a subset of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}})$ (the space of all infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{q}}$) consisting of all solutions of a set of higher order, linear, constant coefficient differential equations. More precisely, there exists a real polynomial matrix R with $\mathtt{q}$

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columns such that $\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}}) \mid R(d/dt)w = 0)\}$. Any such dynamical system Σ is called a linear differential system. The set of all linear differential systems with $\mathtt{q}$ variables is denoted by $\mathcal{L}^{\mathtt{q}}$. Since the behavior $\mathfrak{B}$ of the system Σ is the central item, we will mostly speak about the system $\mathfrak{B} \in \mathcal{L}^{\mathtt{q}}$ (instead of $\Sigma \in \mathcal{L}^{\mathtt{q}}$). Henceforth, in this paper we will suppress the notation 'd/dt', and write Rw instead of $R(d/dt)w$. If a behavior $\mathfrak{B}$ is represented by $Rw = 0$ then we call this a kernel representation of $\mathfrak{B}$, and we often write $\mathfrak{B} = \ker(R)$.

Suppose R has $\mathtt{p}$ rows. Then the kernel representation is said to be minimal if every other kernel representation of $\mathfrak{B}$ has at least $\mathtt{p}$ rows. A given kernel representation $\mathfrak{B} = \ker(R)$ is minimal if and only if the polynomial matrix R has full row rank (see [6], Theorem 3.6.4). The number of rows in any minimal kernel representation of $\mathfrak{B}$ is denoted by $\mathtt{p}(\mathfrak{B})$. This number is called the output cardinality of $\mathfrak{B}$. It corresponds to the number of outputs in any input/output representation of $\mathfrak{B}$. The number of remaining components is called the input cardinality of $\mathfrak{B}$ and is denoted by $\mathtt{m}(\mathfrak{B})$. Thus $\mathtt{m}(\mathfrak{B}) = \mathtt{q} - \mathtt{p}(\mathfrak{B})$.

We now review some facts on elimination. Let $\mathfrak{B} \in \mathcal{L}^{\mathtt{q}}$ with system variable $w = (w_1, w_2)$. Let P_{w_1} denote the projection onto the w_1-component. Then the set $P_{w_1}\mathfrak{B}$ of all w_1 for which there exists w_2 such that $(w_1, w_2) \in \mathfrak{B}$ is again a linear differential system, see [6], section 6.2.2. In this paper we denote $P_{w_1}\mathfrak{B}$ by $\mathfrak{B}_{w_1}$. We call $\mathfrak{B}_{w_1}$ the system obtained by eliminating w_2 from $\mathfrak{B}$.

Next, we review the notions of free and maximally free variables (see [6] and section 2.9 of [2]).

Definition 2.1: Let $\mathfrak{B} \in \mathcal{L}^{\mathtt{q}_1+\mathtt{q}_2}$ with manifest variable (w_1, w_2). We will call w_2 free in $\mathfrak{B}$ if for any choice of $w_2 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}_2})$ there exists w_1 such that $(w_1, w_2) \in \mathfrak{B}$. We call w_2 maximally free if it is free, and we can not enlarge this set with components from w_1 and still continue to have freeness for this enlarged set of variables.

The following result was shown in [6]:

Proposition 2.2: Let $\mathfrak{B} \in \mathcal{L}^{\mathtt{q}_1+\mathtt{q}_2}$ with system variable (w_1, w_2), and let $(R_1 \quad R_2)$ be a minimal representation of $\mathfrak{B}$, then

1) w_2 is free in $\mathfrak{B}$ if and only if R_1 has full row rank;
2) w_2 is maximally free in $\mathfrak{B}$ if and only if R_1 is square and non-singular.

It turns out that in general a behavior $\mathfrak{B}$ has many maximally free sets of variables. However, the number of components of every maximally free set of variables is the same, and is equal to $\mathtt{m}(\mathfrak{B})$, the input cardinality of $\mathfrak{B}$. Also, if in (w_1, w_2) the component w_2 is maximally free, then we call w_2 input and w_1 output of $\mathfrak{B}$.

Definition 2.3: A behavior $\mathfrak{B}$ is called autonomous if $\mathtt{m}(\mathfrak{B}) = 0$. It is called stable if all trajectories in the behavior tend to zero as time tends to $+\infty$.

From [6] if $\mathfrak{B} = \ker(R)$, then $\mathfrak{B}$ is autonomous if and only if R has full column rank and is stable if and only if $R(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}^+$, where $\mathbb{C}^+ = \{s \mid Re(s) \geq 0\}$. Note that a stable behavior is necessarily autonomous.

III. Review of Implementability and Stabilizability

In this section we will briefly recall the notions regular implementability and stabilizability. We will first look at the full interconnection case, i.e., the case when all the plant variables are available for interconnection.

Definition 3.1: Let $\mathcal{P} \in \mathcal{L}^{\mathtt{q}}$ be a plant behavior. A controller for $\mathcal{P}$ is a system behavior $\mathcal{C} \in \mathcal{L}^{\mathtt{q}}$. The full interconnection of $\mathcal{P}$ and $\mathcal{C}$ is defined as the system with behavior $\mathcal{P} \cap \mathcal{C}$. This behavior is called the controlled behavior, and is also an element of $\mathcal{L}^{\mathtt{q}}$. The full interconnection is called regular if $\mathtt{p}(\mathcal{P} \cap \mathcal{C}) = \mathtt{p}(\mathcal{P}) + \mathtt{p}(\mathcal{C})$. In that case we call $\mathcal{C}$ a regular controller.

Definition 3.2: Let $\mathcal{P}, \mathcal{K} \in \mathcal{L}^{\mathtt{q}}$. We say that $\mathcal{K}$ is regularly implementable by full interconnection with respect to $\mathcal{P}$ if there exists a $\mathcal{C} \in \mathcal{L}^q$ such that $\mathcal{K} = \mathcal{P} \cap \mathcal{C}$ and $\mathtt{p}(\mathcal{P} \cap \mathcal{C}) = \mathtt{p}(\mathcal{P}) + \mathtt{p}(\mathcal{C})$.

The following result was shown in [7]:

Proposition 3.3: Let $\mathcal{P} = \ker(R)$ and $\mathcal{K} = \ker(K)$ and R and K have full row rank. Then $\mathcal{K}$ is regularly implementable with respect to $\mathcal{P}$ by full interconnection if and only if there exists a polynomial matrix F of full row rank such that $R = FK$ and, in addition $F(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$.

Next, we will review the issue of stabilization of behaviors. For the definition of stabilizability in a behavioral context (in terms of stable continuation of trajectories), we refer to [6], [11] or [1]. A given plant is stabilizable if and only if we can stabilize it by interconnecting it with a suitable controller, called a stabilizing controller, which is defined as follows [12].

Definition 3.4: Let $\mathcal{P} \in \mathcal{L}^{\mathtt{q}}$. A controller $\mathcal{C} \in \mathcal{L}^{\mathtt{q}}$ is said to be a stabilizing controller if the behavior $\mathcal{P} \cap \mathcal{C}$ is stable and the interconnection is regular.

The following result was shown in [11].

Proposition 3.5: Let $\mathcal{P} \in \mathcal{L}^{\mathtt{q}}$. Then the following statements are equivalent:

1) $\mathcal{P}$ is stabilizable,
2) there exists a stabilizing controller for $\mathcal{P}$,
3) there exists a stable $\mathcal{K} \in \mathcal{L}^{\mathtt{q}}$ that is regularly implementable w.r.t $\mathcal{P}$.

Next we will look at the so called partial interconnection case, in which only a pre-specified subset of the plant variables is available for interconnection. Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{\mathtt{q}+\mathtt{k}}$ be a linear differential system, with system variable (w, c), where w takes its values in $\mathbb{R}^{\mathtt{q}}$ and c in $\mathbb{R}^{\mathtt{k}}$. The variable w should be interpreted as the variable to be controlled, the variable c as the one through which we can interconnect the plant with a controller, called the control variable. Let $\mathcal{C} \in \mathcal{L}^{\mathtt{k}}$ (to be interpreted as a controller behavior) with variable c.

Definition 3.6: The interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ through c is defined as the system behavior $\mathcal{P}_{\text{full}} \wedge_c \mathcal{C} \in \mathcal{L}^{\mathtt{q}+\mathtt{k}}$, given by $\mathcal{P}_{\text{full}} \wedge_c \mathcal{C} = \{(w, c) \mid (w, c) \in \mathcal{P}_{\text{full}} \text{ and } c \in \mathcal{C}\}$. The behavior $\mathcal{P}_{\text{full}} \wedge_c \mathcal{C}$ is called *the full controlled behavior*. The behavior $(\mathcal{P}_{\text{full}} \wedge_c \mathcal{C})_w \in \mathcal{L}^{\mathtt{q}}$ that is obtained by eliminating c from $\mathcal{P}_{\text{full}} \wedge_c \mathcal{C}$ is called *the manifest controlled behavior*. The interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ through c is called *regular* if the output cardinality of the interconnected behavior is the sum of the output cardinalities of the plant and the controller, i.e., $\mathtt{p}(\mathcal{P}_{\text{full}} \wedge_c \mathcal{C}) = \mathtt{p}(\mathcal{P}_{\text{full}}) + \mathtt{p}(\mathcal{C})$. Again, $\mathcal{C}$ is then called a regular controller.

Let $\mathcal{K} \in \mathcal{L}^{\mathtt{q}}$ be a given behavior, which should be interpreted as a 'desired' behavior. A fundamental question is whether this $\mathcal{K}$ can be achieved as controlled behavior by regular interconnection:

Definition 3.7: If there exists a $\mathcal{C} \in \mathcal{L}^{\mathrm{k}}$ such that $\mathcal{K} = (\mathcal{P}_{\text{full}} \wedge_c \mathcal{C})_w$ and $\mathrm{p}(\mathcal{P}_{\text{full}} \wedge_c \mathcal{C}) = \mathrm{p}(\mathcal{P}_{\text{full}}) + \mathrm{p}(\mathcal{C})$, then we call $\mathcal{K}$ regularly implementable by partial interconnection (through c with respect to $\mathcal{P}_{\text{full}}$).

Necessary and sufficient conditions for regular implementability by partial interconnection were obtained in [12] and [1]. An important role is played by the so called hidden behavior:

Definition 3.8: Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{\mathrm{q+k}}$. The hidden behavior of $\mathcal{P}_{\text{full}}$ is the behavior in $\mathcal{L}^{\mathrm{q}}$ defined by $\mathcal{N}_w(\mathcal{P}_{\text{full}}) = \{w \mid (w,0) \in \mathcal{P}_{\text{full}}\}$.

Proposition 3.9: $\mathcal{K} \in \mathcal{L}^{\mathrm{q}}$ is regularly implementable by partial interconnection through c with respect to $\mathcal{P}_{\text{full}}$ if and only if (1.) $\mathcal{N}_w(\mathcal{P}_{\text{full}}) \subseteq \mathcal{K} \subseteq (\mathcal{P}_{\text{full}})_w$ and (2.) $\mathcal{K}$ is regularly implementable with respect to $(\mathcal{P}_{\text{full}})_w$ by full interconnection.

We now recall the definition of stabilizing controller for the partial interconnection case, see [1]:

Definition 3.10: Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{\mathrm{q+k}}$. The controller $\mathcal{C} \in \mathcal{L}^{\mathrm{k}}$ is said to stabilize $\mathcal{P}_{\text{full}}$ through c if the manifest controlled behavior $(\mathcal{P}_{\text{full}} \wedge_c \mathcal{C})_w$ is stable and the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is regular. The controller $\mathcal{C}$ is then called a stabilizing controller.

For the behavioral definition of detectability of a set of system variables from a complementary set of system variables in a given behavior we refer to [6] or [1]. The following result was shown in [1]:

Proposition 3.11: Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{\mathrm{q+k}}$. The following statements are equivalent:

1) there exists a stabilizing controller for $\mathcal{P}_{\text{full}}$;
2) there exists a stable $\mathcal{K} \in \mathcal{L}^{\mathrm{q}}$ that is regularly implementable through c with respect to $\mathcal{P}_{\text{full}}$;
3) $(\mathcal{P}_{\text{full}})_w$ is stabilizable, and in $\mathcal{P}_{\text{full}}$ w is detectable from c.

IV. Problem Formulation

As mentioned above, necessary and sufficient conditions for regular implementability were obtained in [12] and [1]. In these papers the authors deal with controllers without a priori given constraints on their input/output structure, in other words, any (regular) controller from the class of linear differential systems is allowed. Often, by physical considerations, a controller should take information on the plant measurements as its input, and, clearly, such set of measured variables is not allowed to be constrained by the controller. In other words, it is a naturally emerging requirement that a given subset of the plant control variables *should be free in the controller* . In such situations, not all regular controllers are admissible, and, consequently, not all regularly implementable behaviors will be achievable. The problem of regular implementability using controllers in which an a priori given subset of the plant control variables is free was introduced by A. Julius in [3]. Consider the following Example from [3]:

1) Example 4.1: Consider a single tank system as shown in Fig. 1. On top of the tank there is an inlet from which a variable flow of water u gets into the tank. There is an opening at the bottom of the tank connected to a pump through which we can pump in/out water from the tank. The flow which is pumped out of the tank is denoted by y. The tank is also equipped with a sensor which measures the change in volume inside the tank, the measurement of the sensor is denoted by h. The mathematical model of the plant is given by

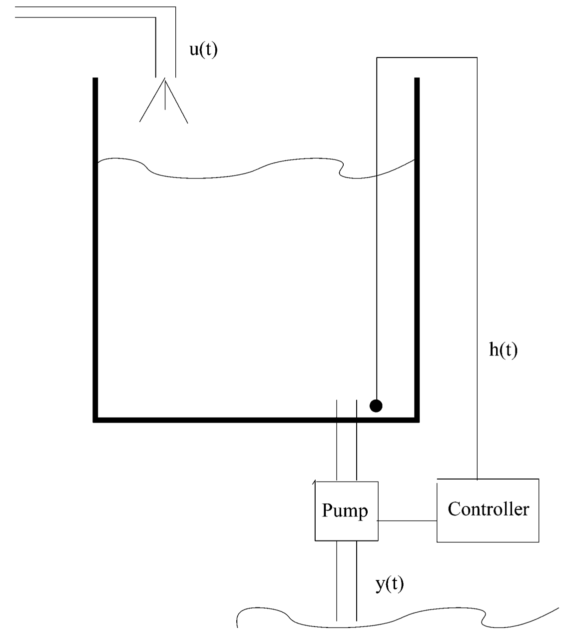

Fig. 1. Single tank system.

$$h = u - y. \quad (1)$$

Consider the following control problem. Given h and y as control variables we want to design a controller such that the level of water inside the tank is constant, i.e $h = 0$ or $y = u$. In other words we aim at perfect tracking of u by y. The problem is mathematically formulated as follows.

Given are $\mathcal{P}_{\text{full}} = \{(u,y,h)) \mid -u + y + h = 0\}$ with plant variable (w,c) where $w = (u,y), c = (y,h)$ and $\mathcal{K} = \{(u,y)) \mid -u + y = 0\}$. From Proposition 3.9 one can check that this $\mathcal{K}$ is regularly implementable by partial interconnection through c with respect to $\mathcal{P}_{\text{full}}$, and a controller which accomplishes this task is given by $\mathcal{C} = \{(h,y) \mid h = 0\}$. Here the variable h is the measurement coming from the system sensor. By physical considerations, this controller is not realizable, as restricting sensor measurement does not make sense practically: h is a control variable which cannot be just put equal to 0 by the controller: it should remain *free* in the controller, become input to the controller. Therefore, even though the given $\mathcal{K}$ is regularly implementable it is not practically realizable.

Motivated by the above, the problems that we solve in this paper may succinctly be formulated as follows: let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{\mathrm{q+k}}$ be a plant behavior, with system variable (w,c). Partition the control variable as $c = (c_1, c_2)$.

2) Problems 1 and 2: Let a desired behavior $\mathcal{K} \in \mathcal{L}^{\mathrm{q}}$ be given. (1.) Find necessary and sufficient conditions such that $\mathcal{K}$ is regularly implementable by a controller in which c_2 is free. (2.) Find necessary and sufficient conditions such that $\mathcal{K}$ is regularly implementable by a controller in which c_2 is maximally free, i.e., in which c_2 is input and c_1 is output.

3) Problems 3 and 4: (3.) Find necessary and sufficient conditions for the existence of a stabilizing controller in which c_2

is free. (4.) Find necessary and sufficient conditions for the existence of a stabilizing controller in which c_2 is maximally free.

V. Regular Implementability Using Controllers With Pre-Specified Input/Output Structure

In this section we study Problems (1) and (2) above. We study these problems for the full interconnection case first.

A. Full Interconnection

Let $\mathcal{P}$, $\mathcal{K} \in \mathcal{L}^{\mathrm{q}_1+\mathrm{q}_2}$ with plant variable (w_1, w_2). In the full interconnection case, a controller is a system $\mathcal{C} \in \mathcal{L}^{\mathrm{q}_1+\mathrm{q}_2}$ acting on the entire plant variable (w_1, w_2). We impose that the variable w_2 should be free in the controller $\mathcal{C}$, and we want to find conditions on the desired behavior $\mathcal{K}$ to be regularly implementable by such controller $\mathcal{C}$. In the following theorem, let $(\mathcal{K})_{w_2}$ denote the projection of $\mathcal{K}$ onto the variable w_2. We have:

Theorem 5.1: Let $\mathcal{P}$, $\mathcal{K} \in \mathcal{L}^{\mathrm{q}_1+\mathrm{q}_2}$ with plant variable (w_1, w_2). Then $\mathcal{K}$ is regularly implementable by full interconnection with respect to $\mathcal{P}$ using a controller $\mathcal{C}$ in which w_2 is free if and only if the following conditions hold:

1) $\mathcal{K}$ is regularly implementable by full interconnection with respect to $\mathcal{P}$;
2) $\mathrm{p}((\mathcal{K})_{w_2}) \leq \mathrm{p}(\mathcal{P})$.

Before proving this theorem we will establish some results that are useful in the proof. Associated with $\mathcal{K} \in \mathcal{L}^{\mathrm{q}_1+\mathrm{q}_2}$ with plant variable (w_1, w_2), we define $\mathcal{N}_{w_1}(\mathcal{K}) = \{w_1 \mid (w_1, 0) \in \mathcal{K}\}$. Then we have the following lemma:

Lemma 5.2: Let $\mathcal{K} \in \mathcal{L}^{\mathrm{q}_1+\mathrm{q}_2}$ with system variable (w_1, w_2). Then we have $\mathrm{p}((\mathcal{K})_{w_2}) = \mathrm{p}(\mathcal{K}) - \mathrm{p}(\mathcal{N}_{w_1}(\mathcal{K}))$.

Proof: Let $(\,K_1 \quad K_2\,)$ be a minimal representation of $\mathcal{K}$. Then $\mathcal{N}_{w_1}(\mathcal{K}) = \ker(K_1)$. From Lemma 8 of [1] we have $\mathrm{p}((\mathcal{K})_{w_2}) = \mathrm{rank}(\,K_1 \quad K_2\,) - \mathrm{rank}(K_1) = \mathrm{p}(\mathcal{K}) - \mathrm{p}(\mathcal{N}_{w_1}(\mathcal{K}))$. □

We also use an important result obtained as Lemma 4.73 in [3] to prove our Theorem 5.1 (see also [5]). This result is stated as a lemma here.

Lemma 5.3: Let C and M be polynomial matrices with the same number of columns. There exists a polynomial matrix V such that $C + VM$ has full row rank if and only if

$$\mathrm{rank}\begin{pmatrix} M \\ C \end{pmatrix} \geq \mathrm{rowdim}(C).$$

Using the above lemmas we now give a proof of Theorem 5.1.

Proof of Theorem 5.1: (if) Let $R = (\,R_1 \quad R_2\,)$ and $K = (\,K_1 \quad K_2\,)$ give minimal kernel representations of the behaviors $\mathcal{P}$ and $\mathcal{K}$, respectively. From Proposition 3.3, condition 1 implies that there exists F such that $R = FK$ and $F(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. Take W such that $\begin{pmatrix} F \\ W \end{pmatrix}$ forms a unimodular matrix. From [7], Theorem 11, WK has full row rank, $\ker(WK)$ regularly implements $\mathcal{K}$, and a parametrization of all controllers which regularly implement $\mathcal{K}$ is given by $GR + UWK$, where G is an arbitrary polynomial matrix, and U is unimodular. From the above arguments, $\ker\begin{pmatrix} R_1 & R_2 \\ WK_1 & WK_2 \end{pmatrix} = \ker(\,K_1 \quad K_2\,)$, which implies $\mathcal{N}_{w_1}(\mathcal{K}) = \ker\begin{pmatrix} R_1 \\ WK_1 \end{pmatrix}$. Therefore

$$\mathrm{p}(\mathcal{N}_{w_1}(\mathcal{K})) = \mathrm{rank}\begin{pmatrix} R_1 \\ WK_1 \end{pmatrix}. \tag{2}$$

Since $\ker(WK)$ regularly implements $\mathcal{K}$, we have $\mathrm{p}(\mathcal{K}) = \mathrm{p}(\mathcal{P}) + \mathrm{rank}(WK) = \mathrm{p}(\mathcal{P}) + \mathrm{rowdim}(WK_1)$, which implies $\mathrm{p}(\mathcal{P}) = \mathrm{p}(\mathcal{K}) - \mathrm{rowdim}(WK_1)$. Condition 2 together with lemma 5.2 imply that $\mathrm{p}(\mathcal{K}) - \mathrm{p}(\mathcal{N}_{w_1}(\mathcal{K})) \leq \mathrm{p}(\mathcal{K}) - \mathrm{rowdim}(WK_1)$. This implies $\mathrm{p}(\mathcal{N}_{w_1}(\mathcal{K})) \geq \mathrm{rowdim}(WK_1)$, so $\mathrm{rank}\begin{pmatrix} R_1 \\ WK_1 \end{pmatrix} \geq \mathrm{rowdim}(WK_1)$. Using this last inequality, by Lemma 5.3 there exists a G_0 such that $G_0R_1 + WK_1$ has full row rank. Define $\mathcal{C}_0 = \ker(G_0R + WK)$. Then $\mathcal{C}_0$ is a controller in which w_2 is free and that regularly implements $\mathcal{K}$.

(only if) Let $C = (\,C_1 \quad C_2\,)$ be a minimal kernel representation of a controller $\mathcal{C}$ in which w_2 is free and which regularly implements $\mathcal{K}$. Then $\mathcal{K} = \ker\begin{pmatrix} R_1 & R_2 \\ C_1 & C_2 \end{pmatrix}$ and $\mathcal{N}_{w_1}(\mathcal{K}) = \ker\begin{pmatrix} R_1 \\ C_1 \end{pmatrix}$. We know that w_2 is free in $\mathcal{C}$ if and only if C_1 has full row rank, which implies $\mathrm{p}(\mathcal{P}) = \mathrm{rank}(C_1)$. Therefore from regular implementability we have $\mathrm{p}(\mathcal{K}) = \mathrm{p}(\mathcal{P}) + \mathrm{rank}(C_1)$. From Lemma 5.2 we have $\mathrm{p}((\mathcal{K})_{w_2}) = \mathrm{p}(\mathcal{K}) - \mathrm{p}(\mathcal{N}_{w_1}(\mathcal{K})) = \mathrm{p}(\mathcal{P}) + \mathrm{rank}(C_1) - \mathrm{rank}\begin{pmatrix} R_1 \\ C_1 \end{pmatrix}$. This implies $\mathrm{p}((\mathcal{K})_{w_2}) \leq \mathrm{p}(\mathcal{P})$. □

We now derive conditions on $\mathcal{K}$ to be regularly implementable by a controller $\mathcal{C}$ in which w_2 is *maximally free* , equivalently, in $\mathcal{C}$ w_2 is input and w_1 is output.

Theorem 5.4: Let $\mathcal{P}$, $\mathcal{K} \in \mathcal{L}^{\mathrm{q}_1+\mathrm{q}_2}$ with plant variable (w_1, w_2). Then $\mathcal{K}$ is regularly implementable by full interconnection with respect to $\mathcal{P}$ using a controller $\mathcal{C}$ in which w_2 is input and w_1 is output if and only if the following conditions hold:

1) $\mathcal{K}$ is regularly implementable by full interconnection with respect to $\mathcal{P}$;
2) $\mathrm{p}((\mathcal{K})_{w_2}) = \mathrm{p}(\mathcal{P})$;
3) $\mathrm{p}(\mathcal{K}) = \mathrm{q}_1 + \mathrm{p}(\mathcal{P})$.

Proof: (only if) Let $K = (\,K_1 \quad K_2\,)$, $R = (\,R_1 \quad R_2\,)$ give minimal kernel representations of the behaviors $\mathcal{K}$ and $\mathcal{P}$, and let $C = (\,C_1 \quad C_2\,)$ be a minimal kernel representation of $\mathcal{C}$ which regularly implements $\mathcal{K}$ and in which w_2 is maximally free. Note that as $\ker\begin{pmatrix} R_1 & R_2 \\ C_1 & C_2 \end{pmatrix} = \ker(\,K_1 \quad K_2\,)$, from [1], lemma 8 (see also [2], lemma 2.9.5) we have $\mathrm{p}((\mathcal{K})_{w_2}) = \mathrm{rank}\begin{pmatrix} R_1 & R_2 \\ C_1 & C_2 \end{pmatrix} - \mathrm{rank}\begin{pmatrix} R_1 \\ C_1 \end{pmatrix} = \mathrm{p}(\mathcal{P}) + \mathrm{p}(\mathcal{C}) - \mathrm{rank}(C_1) = \mathrm{p}(\mathcal{P})$. From regularity of the interconnection we have $\mathrm{p}(\mathcal{K}) = \mathrm{p}(\mathcal{P}) + \mathrm{p}(\mathcal{C})$. As $\mathrm{p}(\mathcal{C}) = \mathrm{q}_1$ we get $\mathrm{p}(\mathcal{K}) = \mathrm{p}(\mathcal{P}) + \mathrm{q}_1$.

(if) Let $\mathcal{K} = \ker(\,K_1 \quad K_2\,)$, $\mathcal{P} = \ker(\,R_1 \quad R_2\,)$ be minimal representations. Then, using Proposition 3.3, condition 1 implies that there exists a matrix F such that $(\,FK_1 \quad FK_2\,) = (\,R_1 \quad R_2\,)$, and $F(\lambda)$ has full row rank for

all $\lambda \in \mathbb{C}$. Choose W such that $\begin{pmatrix} F \\ W \end{pmatrix}$ forms a unimodular matrix. Then, again by [7], Theorem 11, a parametrization of all controllers which regularly implement $\mathcal{K}$ with respect to $\mathcal{P}$ is given by $(C_1 \quad C_2) = (GR_1 + UWK_1 \quad GR_2 + UWK_2)$, where G is an arbitrary polynomial matrix, and U is unimodular. Condition 2 along with Theorem 5.1 implies that there exists a G such that $GR_1 + WK_1$ has full row rank. From condition 3 we have $\mathtt{p}(\mathcal{K}) - \mathtt{p}(\mathcal{P}) = \mathtt{q}_1$. This is equivalent to $\operatorname{rank}\begin{pmatrix} R_1 & R_2 \\ GR_1 + WK_1 & GR_2 + WK_2 \end{pmatrix} - \operatorname{rank}(R_1 \quad R_2) = \operatorname{coldim}(GR_1 + WK_1)$, which in turn is equivalent to $\operatorname{rank}(GR_1 + WK_1 \quad GR_2 + WK_2) = \operatorname{coldim}(GR_1 + WK_1)$. Therefore $\operatorname{rowdim}(GR_1 + WK_1) = \operatorname{coldim}(GR_1 + WK_1)$ which implies that $GR_1 + WK_1$ is square. We conclude that condition 2 and 3 together imply that there exists a matrix G such that $GR_1 + WK_1$ is square and nonsingular. Define $\mathcal{C} = \ker(GR_1 + WK_1 \quad GR_2 + WK_2)$. Then $\mathcal{C}$ is a controller in which w_2 is input and w_1 is output and which regularly implements $\mathcal{K}$ by full interconnection with respect to $\mathcal{P}$. □

Remark 5.5: In the special case that $\mathcal{K}$ is autonomous we have $\mathtt{p}((\mathcal{K})_{w_2}) = \mathtt{q}_2$. In that case condition 2 of Theorem 5.1 becomes $\mathtt{q}_2 \leq \mathtt{p}(\mathcal{P})$ and conditions 2 and 3 of Theorem 5.4 reduce to the single condition $\mathtt{p}(\mathcal{P}) = \mathtt{q}_2$. Thus, $\mathcal{K}$ is regularly implementable using a controller with w_2 free (maximally free), if and only if it is regularly implementable and the number of components of w_2 does not exceed (is equal to) the output cardinality of the plant. It is remarkable that these conditions do not involve *which* components, but only the *number* of components of that should be free.

B. Partial Interconnection

We now deal with Problems (1) and (2) as formulated in Section IV. We will solve these problems by reduction to the full interconnection case. In the sequel, the interconnected behavior $\mathcal{P}_{\text{full}} \wedge_w \mathcal{K}$ plays an important role. In fact, the behavior $(\mathcal{P}_{\text{full}} \wedge_w \mathcal{K})_c$ obtained from this interconnection by eliminating the variable w is often called the canonical controller, see [4], [5], [9]. The following proposition was obtained in [10] (see also [8]).

Proposition 5.6: Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{\mathtt{q}+\mathtt{k}}$ with system variable (w, c). Then $\mathcal{K} \in \mathcal{L}^{\mathtt{q}}$ is regularly implementable by partial interconnection through c with respect to $\mathcal{P}_{\text{full}}$ if and only if the following two conditions hold:

1) $\mathcal{K}$ is implementable by partial interconnection through c with respect to $\mathcal{P}_{\text{full}}$;
2) $(\mathcal{P}_{\text{full}} \wedge_w \mathcal{K})_c$ is regularly implementable by full interconnection with respect to $(\mathcal{P}_{\text{full}})_c$.

The next result was obtained in [10], corollary 14 (see also [8], proposition 1).

Proposition 5.7: Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{\mathtt{q}+\mathtt{k}}$ with system variable (w, c). Let $\mathcal{K} \in \mathcal{L}^{\mathtt{q}}$. If $\mathcal{C} \in \mathcal{L}^{\mathtt{k}}$ regularly implements $\mathcal{K}$ by partial interconnection (through c with respect to $\mathcal{P}_{\text{full}}$), then $\mathcal{C}' = \mathcal{C} + \{c \mid (0, c) \in \mathcal{P}_{\text{full}}\}$ regularly implements $(\mathcal{P}_{\text{full}} \wedge_w \mathcal{K})_c$ with respect to $(\mathcal{P}_{\text{full}})_c$ by full interconnection. Moreover any controller $\mathcal{C} \in \mathcal{L}^{\mathtt{k}}$ which regularly implements $(\mathcal{P}_{\text{full}} \wedge_w \mathcal{K})_c$ with respect to $(\mathcal{P}_{\text{full}})_c$ by full interconnection regularly implements $\mathcal{K}$ by partial interconnection (through c with respect to $\mathcal{P}_{\text{full}}$).

The following theorem now provides a solution to Problem (1):

Theorem 5.8: Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{\mathtt{q}+\mathtt{k}}$ with system variable (w, c). Partition $c = (c_1, c_2)$. Then $\mathcal{K} \in \mathcal{L}^{\mathtt{q}}$ is regularly implementable by partial interconnection through c with respect to $\mathcal{P}_{\text{full}}$ using a controller in which c_2 is free if and only if the following conditions hold:

1) $\mathcal{K}$ is regularly implementable by partial interconnection through c with respect to $\mathcal{P}_{\text{full}}$;
2) $\mathtt{p}((\mathcal{P}_{\text{full}} \wedge_w \mathcal{K})_{c_2}) \leq \mathtt{p}((\mathcal{P}_{\text{full}})_c)$.

Proof: (only if) From Proposition 5.7, if $\mathcal{C}$ regularly implements $\mathcal{K}$ by partial interconnection, then $\mathcal{C}' = \mathcal{C} + \{c \mid (0, c) \in \mathcal{P}_{\text{full}}\}$ regularly implements $(\mathcal{P}_{\text{full}} \wedge_w \mathcal{K})_c$ with respect to $(\mathcal{P}_{\text{full}})_c$ by full interconnection. We note that $\mathcal{C} \subseteq \mathcal{C}'$. Therefore if $C = (C_1 \quad C_2)$ and $C' = (C'_1 \quad C'_2)$ are minimal kernel representations of $\mathcal{C}$ and $\mathcal{C}'$ respectively, then there exists a polynomial matrix F such that $C' = FC$. As C and C' have full row rank, F also has full row rank. If c_2 is free in $\mathcal{C}$ then it is also free in $\mathcal{C}'$ (since if C_1 has full row rank then $C'_1 = FC_1$ will also have full row rank). As $\mathcal{C}'$ regularly implements $(\mathcal{P}_{\text{full}} \wedge_w \mathcal{K})_c$ through full interconnection with respect to $(\mathcal{P}_{\text{full}})_c$, from Theorem 5.1 it directly follows that $\mathtt{p}((\mathcal{P}_{\text{full}} \wedge_w \mathcal{K})_{c_2}) \leq \mathtt{p}((\mathcal{P}_{\text{full}})_c)$.

(if) Using Theorem 5.1 and Proposition 5.6, condition 1 and 2 together imply that there exists a controller $\mathcal{C}$ in which c_2 is free and which regularly implements $(\mathcal{P}_{\text{full}} \wedge_w \mathcal{K})_c$ with respect to $(\mathcal{P}_{\text{full}})_c$ through full interconnection. From Proposition 5.7 the same $\mathcal{C}$ regularly implements $\mathcal{K}$ by partial interconnection (through c with respect to $\mathcal{P}_{\text{full}}$). □

Remark 5.9: Let $\mathcal{P} = \ker(R_1 \quad R_{21} \quad R_{22})$ with system variable (w, c_1, c_2) and assume that $\mathcal{K} = \ker(K)$ be autonomous. Then, using [1], Lemma 8 we have $\mathtt{p}((\mathcal{P}_{\text{full}} \wedge_w \mathcal{K})_{c_2}) = \operatorname{rank}\begin{pmatrix} R_1 & R_{21} & R_{22} \\ K & 0 & 0 \end{pmatrix} - \operatorname{rank}\begin{pmatrix} R_1 & R_{21} \\ K & 0 \end{pmatrix} = \operatorname{rank}(R_{21} \quad R_{22}) - \operatorname{rank}(R_{21}) = \mathtt{p}(\mathcal{N}_{(c_1,c_2)}(\mathcal{P}_{\text{full}})) - \mathtt{p}(\mathcal{N}_{c_1}(\mathcal{P}_{\text{full}}))$.

Then condition 2 of Theorem 5.8 becomes

$$\mathtt{p}(\mathcal{N}_{(c_1,c_2)}(\mathcal{P}_{\text{full}})) - \mathtt{p}(\mathcal{N}_{c_1}(\mathcal{P}_{\text{full}})) \leq \mathtt{p}(\mathcal{P}_{\text{full}}) - \mathtt{p}(\mathcal{N}_w(\mathcal{P}_{\text{full}})). \tag{3}$$

In other words, a given autonomous $\mathcal{K}$ is regularly implementable using a controller in which c_2 is free if and only if it is regularly implementable, and the inequality (3) holds. Note that, surprisingly, (3) is a condition only in terms of $\mathcal{P}_{\text{full}}$ and the partition (c_1, c_2) of the control variable, and is independent of $\mathcal{K}$.

Our next result provides a solution to Problem (2):

Theorem 5.10: Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{\mathtt{q}+\mathtt{k}}$ with system variable (w, c). Let $\mathcal{K} \in \mathcal{L}^{\mathtt{q}}$. Partition $c = (c_1, c_2)$ with c_1 of size $\mathtt{k}_1$, c_2 of size $\mathtt{k}_2$ and $\mathtt{k}_1 + \mathtt{k}_2 = \mathtt{k}$. Consider the following three conditions:

1) $\mathcal{K}$ is regularly implementable by partial interconnection through c with respect to $\mathcal{P}_{\text{full}}$;
2) $\mathtt{p}((\mathcal{P}_{\text{full}} \wedge_w \mathcal{K})_{c_2}) = \mathtt{p}((\mathcal{P}_{\text{full}})_c)$;
3) $\mathtt{p}((\mathcal{P}_{\text{full}} \wedge_w \mathcal{K})_c) = \mathtt{k}_1 + \mathtt{p}((\mathcal{P}_{\text{full}})_c)$.

If 1, 2 and 3 hold then $\mathcal{K}$ is regularly implementable by means of a controller in which c_2 is maximally free. If $\{c \mid (0,c) \in \mathcal{P}_{\text{full}}\}$ is autonomous, then 1, 2, and 3 are also necessary for the existence of a controller $\mathcal{C}$ that implements $\mathcal{K}$ and in which c_2 is maximally free.

Before proving the theorem we will establish a result that will be useful in the proof. Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{\mathtt{q}+\mathtt{k}}$ with plant variable (w,c), and let $\mathcal{C}$ regularly implement $\mathcal{K} \in \mathcal{L}^{q}$ through c with respect to $\mathcal{P}_{\text{full}}$. Denote $\mathcal{N}_c(\mathcal{P}_{\text{full}}) = \{c \mid (0,c) \in \mathcal{P}_{\text{full}}\}$ and define $\mathcal{C}' = \mathcal{C} + \mathcal{N}_c(\mathcal{P}_{\text{full}})$. Then we have the following lemma:

Lemma 5.11: Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{\mathtt{q}+\mathtt{k}}$ with system variable (w,c). Assume $\mathcal{N}_c(\mathcal{P}_{\text{full}})$ is autonomous. Let $\mathcal{C} \in \mathcal{L}^{k}$ regularly implement $\mathcal{K} \in \mathcal{L}^{\mathtt{q}}$ through c with respect to $\mathcal{P}_{\text{full}}$. If C and C' are minimal kernel representations of $\mathcal{C}$ and $\mathcal{C}'$ respectively, then there exists a square nonsingular polynomial matrix F such that $C' = FC$.

Proof: It is evident that $\mathcal{C} \subseteq \mathcal{C}'$, therefore there exists a polynomial matrix F such that $C' = FC$. As C, C' have full row rank F also has full row rank. Let $(\, R_1 \quad R_2 \,)$ give a minimal kernel representation of $\mathcal{P}_{\text{full}}$. Then $\mathcal{N}_c(\mathcal{P}_{\text{full}}) = \ker(R_2)$, and is autonomous if and only if R_2 has full column rank. As both $\mathcal{C}$ and $\mathcal{C}'$ regularly implement $\mathcal{K}$ by partial interconnection with respect to $\mathcal{P}_{\text{full}}$ (see [10], Corollary 14 or [8] Proposition 1) we have $\mathtt{p}(\mathcal{K}) = \mathtt{p}((\mathcal{P}_{\text{full}} \wedge_c \mathcal{C})_w) = \mathtt{p}((\mathcal{P}_{\text{full}} \wedge_c \mathcal{C}')_w)$. Also from [1], lemma 8, $\mathtt{p}((\mathcal{P}_{\text{full}} \wedge_c \mathcal{C})_w) = \operatorname{rank}\begin{pmatrix} R_1 & R_2 \\ 0 & C \end{pmatrix} - \operatorname{rank}\begin{pmatrix} R_2 \\ C \end{pmatrix}$, which is equivalent to $\mathtt{p}((\mathcal{P}_{\text{full}} \wedge_c \mathsf{C})_w) = \operatorname{rank}(\, R_1 \quad R_2 \,) + \operatorname{rank}(C) - \operatorname{rank}(R_2)$. Similarly, $\mathtt{p}((\mathcal{P}_{\text{full}} \wedge_c \mathcal{C}')_w) = \operatorname{rank}(\, R_1 \quad R_2 \,) + \operatorname{rank}(C') - \operatorname{rank}(R_2)$. Putting things together we get $\operatorname{rank}(C) = \operatorname{rank}(C')$. This implies $\operatorname{rowdim}(F) = \operatorname{coldim}(F)$. $\square$

Using the above lemma we now prove Theorem 5.10.

Proof of Theorem 5.10: (only if) From Proposition 5.7, if $\mathcal{C}$ regularly implements $\mathcal{K}$ then $\mathcal{C}' = \mathcal{C} + \{c \mid (0,c) \in \mathcal{P}_{\text{full}}\}$ regularly implements $(\mathcal{P}_{\text{full}} \wedge_w \mathcal{K})_c$ with respect to $(\mathcal{P}_{\text{full}})_c$. Let $C = (\, C_1 \quad C_2 \,)$ and $C' = (\, C'_1 \quad C'_2 \,)$ be minimal kernel representations of $\mathcal{C}$ and $\mathcal{C}'$ respectively. Then from lemma 5.11 there exists a polynomial matrix F which is square and nonsingular such that $(\, C'_1 \quad C'_2 \,) = (\, FC_1 \quad FC_2 \,)$. If c_2 is maximally free in $\mathcal{C}$ then it is also maximally free in $\mathcal{C}'$ (since if C_1 is square and nonsingular then the same will hold for FC_1). Conditions 2 and 3 of the Theorem directly follow from Theorem 5.4.

(if) From condition 1 and using Proposition 5.6, $(\mathcal{P}_{\text{full}} \wedge_w \mathcal{K})_c$ is regularly implementable with respect to $(\mathcal{P}_{\text{full}})_c$ by full interconnection. Conditions 2 and 3 imply that there exists a controller $\tilde{\mathcal{C}}$ which regularly implements $(\mathcal{P}_{\text{full}} \wedge_w \mathcal{K})_c$ with respect to $(\mathcal{P}_{\text{full}})_c$ by full interconnection and c_2 is maximally free in $\tilde{\mathcal{C}}$. From Proposition 5.7, the same $\tilde{\mathcal{C}}$ regularly implements $\mathcal{K}$ by partial interconnection (through c with respect to $\mathcal{P}_{\text{full}}$). $\square$

Remark 5.12: In the special case that $\mathcal{K}$ is autonomous, conditions 2 and 3 in Theorem 5.10 become

1) $\mathtt{p}(\mathcal{N}_c(\mathcal{P}_{\text{full}})) - \mathtt{p}(\mathcal{N}_{c_1}(\mathcal{P}_{\text{full}})) = \mathtt{p}((\mathcal{P}_{\text{full}})_c)$;
2) $\mathtt{p}(\mathcal{N}_c(\mathcal{P}_{\text{full}})) = \mathtt{k}_1 + \mathtt{p}((\mathcal{P}_{\text{full}})_c)$.

Moreover, if $\mathcal{N}_c(\mathcal{P}_{\text{full}})$ also happens to be autonomous, then these conditions reduce to the single condition $\mathtt{p}((\mathcal{P}_{\text{full}})_c) = \mathtt{k}_2$. Hence we get the following: if $\mathcal{P}_{\text{full}}$ is such that $\mathcal{N}_c(\mathcal{P}_{\text{full}})$ is autonomous, then a given autonomous $\mathcal{K}$ is regularly implementable using a controller with c_2 input and c_1 output if and only if it is regularly implementable and $\mathtt{p}((\mathcal{P}_{\text{full}})_c) = \mathtt{k}_2$, the number of components of c_2.

VI. Stabilization Using Controllers With *a Priori* Input/Output Structure

In this section we study Problems (3) and (4) as formulated in Section IV. Again, we consider the full interconnection case first.

A. Full Interconnection

Theorem 6.1: Let $\mathcal{P} \in \mathcal{L}^{\mathtt{q}_1+\mathtt{q}_2}$ with plant variable (w_1,w_2). There exists a stabilizing controller $\mathcal{C} \in \mathcal{L}^{\mathtt{q}_1+\mathtt{q}_2}$ in which w_2 is free if and only if $\mathcal{P}$ is stabilizable and $\mathtt{q}_2 \leq \mathtt{p}(\mathcal{P})$.

Proof: (only if) If there exists a stabilizing controller in which w_2 is free then by Proposition 3.5 there exists a stable $\mathcal{K}$ which is regularly implementable by full interconnection with respect to $\mathcal{P}$ using a controller in which w_2 is free. Stabilizability follows from Proposition 3.5, while the inequality $\mathtt{q}_2 \leq \mathtt{p}(\mathcal{P})$ follows from Theorem 5.1 and Remark 5.5.

(if) If $\mathcal{P}$ is stabilizable then by Proposition 3.5 there exists a stable $\mathcal{K}$ which is regularly implementable. Condition $\mathtt{q}_2 \leq \mathtt{p}(\mathcal{P})$ along with Theorem 5.1 and Remark 5.5 implies that this $\mathcal{K}$ is indeed regularly implementable by a controller in which w_2 is free. $\square$

The following theorem gives necessary and sufficient conditions in the full interconnection case for the existence of a stabilizing controller in which a given subset of the control variables is maximally free.

Theorem 6.2: Let $\mathcal{P} \in \mathcal{L}^{\mathtt{q}_1+\mathtt{q}_2}$ with plant variable (w_1,w_2). There exists a stabilizing controller $\mathcal{C} \in \mathcal{L}^{\mathtt{q}_1+\mathtt{q}_2}$ for which w_2 is input and w_1 is output if and only if $\mathcal{P}$ is stabilizable and $\mathtt{q}_2 = \mathtt{p}(\mathcal{P})$.

Proof: A proof of this theorem follows is similar to the proof of Theorem 6.1, and again uses Theorem 5.4 and Remark 5.5. $\square$

B. Partial Interconnection

The following theorem provides a solution to Problem (3):

Theorem 6.3: Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{\mathtt{q}+\mathtt{k}}$ with system variable (w,c). Partition $c = (c_1,c_2)$. There exists a stabilizing controller $\mathcal{C} \in \mathcal{L}^{\mathtt{k}}$ in which c_2 is free if and only if

1) $(\mathcal{P}_{\text{full}})_w$ is stabilizable, and in $\mathcal{P}_{\text{full}}$ w is detectable from c;
2) $\mathtt{p}(\mathcal{N}_c(\mathcal{P}_{\text{full}})) - \mathtt{p}(\mathcal{N}_{c_1}(\mathcal{P}_{\text{full}})) \leq \mathtt{p}((\mathcal{P}_{\text{full}})_c)$.

Proof: (only if) If there exists a stabilizing controller in which c_2 is free then from Proposition 3.11 there exists a stable $\mathcal{K}$ which is regularly implementable by partial interconnection with respect to $\mathcal{P}_{\text{full}}$ using a controller in which c_2 is free. Condition 1 directly follows from Proposition 3.11, while the inequality $\mathtt{p}(\mathcal{N}_c(\mathcal{P}_{\text{full}})) - \mathtt{p}(\mathcal{N}_{c_1}(\mathcal{P}_{\text{full}})) \leq \mathtt{p}((\mathcal{P}_{\text{full}})_c)$ follows from Theorem 5.8 and Remark 5.9.

(if) If $(\mathcal{P}_{\text{full}})_w$ is stabilizable and in $\mathcal{P}_{\text{full}}$ w is detectable from c then from Proposition 3.11 there exists a stable $\mathcal{K}$ which is regularly implementable by partial interconnection with respect to $\mathcal{P}_{\text{full}}$. Condition 2 of the theorem along with Remark 5.9

and Theorem 5.8 implies that this $\mathcal{K}$ is indeed regularly implementable by a controller in which c_2 is free. $\square$

Finally, we give a solution to Problem (4):

Theorem 6.4: Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{\mathtt{q}+\mathtt{k}}$ with system variable (w, c). Partition $c = (c_1, c_2)$ with c_1 size $\mathtt{k}_1$, c_2 size $\mathtt{k}_2$ and $\mathtt{k}_1 + \mathtt{k}_2 = \mathtt{k}$. Consider the following conditions

1) $(\mathcal{P}_{\text{full}})_w$ is stabilizable, and in $\mathcal{P}_{\text{full}}$ w is detectable from c;
2) $\mathtt{p}(\mathcal{N}_c(\mathcal{P}_{\text{full}})) - \mathtt{p}(\mathcal{N}_{c_1}(\mathcal{P}_{\text{full}})) = \mathtt{p}((\mathcal{P}_{\text{full}})_c)$;
3) $\mathtt{p}(\mathcal{N}_c(\mathcal{P}_{\text{full}})) = \mathtt{k}_1 + \mathtt{p}((\mathcal{P}_{\text{full}})_c)$.

If condition 1,2 and 3 hold, then there exists a stabilizing controller $\mathcal{C} \in \mathcal{L}^{\mathtt{k}_1+\mathtt{k}_2}$ for which c_2 is input and c_1 is output. If $\mathcal{N}_c(\mathcal{P}_{\text{full}})$ is autonomous then these conditions are also necessary, and conditions 2 and 3 reduce to the single condition $\mathtt{p}((\mathcal{P}_{\text{full}})_c) = \mathtt{k}_2$.

Proof: (only if) If there exists a stabilizing controller in which c_2 is input, then there exists a stable $\mathcal{K}$ which is regularly implementable by partial interconnection with respect to $\mathcal{P}_{\text{full}}$ using a controller in which c_2 is input. Condition 1 directly follows from Proposition 3.11. If $\mathcal{N}_c(\mathcal{P}_{\text{full}})$ is autonomous then from Theorem 5.10 and Remark 5.12 we have $\mathtt{p}((\mathcal{P}_{\text{full}})_c) = \mathtt{k}_2$.

(if) If $(\mathcal{P}_{\text{full}})_w$ is stabilizable and in $\mathcal{P}_{\text{full}}$ w is detectable from c then from Proposition 3.11, there exists a stable $\mathcal{K}$ which is regularly implementable by partial interconnection with respect to $\mathcal{P}_{\text{full}}$. Conditions 2 and 3 along with Remark 5.12 and Theorem 5.10 imply that this $\mathcal{K}$ is indeed regularly implementable by a controller in which c_2 is input and c_1 is output. $\square$

VII. Worked Out Examples

In order to illustrate the theory developed in this paper, we now present some worked-out examples.

1) Example 7.1: Let $\mathcal{P}_{\text{full}}$ with manifest variable $w = (w_1, w_2)$ and control variable $c = (c_1, c_2, c_3)$ be represented by the equations $w_1 + \dot{w}_2 + c_2 + \dot{c}_3 = 0$, $w_2 + c_1 + c_2 + c_3 = 0$, $\dot{c}_2 + c_3 = 0$. Clearly $\mathtt{p}(\mathcal{P}_{\text{full}}) = 3$, and $(\mathcal{P}_{\text{full}})_w = \mathfrak{C}(\mathbb{R}, \mathbb{R}^2)$. For $\mathcal{K}$ take the behavior represented by $w_1 + \dot{w}_2 = 0$. $\mathcal{K}$ is regularly implementable through (c_1, c_2, c_3) w.r.t. $\mathcal{P}_{\text{full}}$. We have $R(\xi) = \begin{pmatrix} 1 & \xi & 0 & 1 & \xi \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \xi & 1 \end{pmatrix}$, $K(\xi) = (1 \quad \xi)$. $\mathtt{p}(\mathcal{P}_{\text{full}})_c = \operatorname{rank}(R(\xi)) - \operatorname{rank}\begin{pmatrix} 1 & \xi \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = 1$. Now $\mathcal{P}_{\text{full}} \wedge_w \mathcal{K} = \ker(P(d/dt))$, where $P(\xi) = \begin{pmatrix} 1 & \xi & 0 & 1 & \xi \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \xi & 1 \\ 1 & \xi & 0 & 0 & 0 \end{pmatrix}$. Then $\mathtt{p}((\mathcal{P}_{\text{full}} \wedge_w \mathcal{K})_{(c_2,c_3)}) = \operatorname{rank}(P(\xi)) - \operatorname{rank}\begin{pmatrix} 1 & \xi & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & \xi & 0 \end{pmatrix} = 2$ and $\mathtt{p}((\mathcal{P}_{\text{full}} \wedge_w \mathcal{K})_{(c_1,c_2)}) = \operatorname{rank}(P(\xi)) - \operatorname{rank}\begin{pmatrix} 1 & \xi & \xi \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & \xi & 0 \end{pmatrix} = 1$. From these calculations it is evident that $\mathtt{p}((\mathcal{P}_{\text{full}} \wedge_w \mathcal{K})_{(c_1,c_2)}) = \mathtt{p}(\mathcal{P}_{\text{full}})_c$ and $\mathtt{p}((\mathcal{P}_{\text{full}} \wedge_w \mathcal{K})_{(c_2,c_3)}) > \mathtt{p}(\mathcal{P}_{\text{full}})_c$. From Theorem 5.8 we conclude that $\mathcal{K}$ is regularly implementable using a controller in which (c_1, c_2) is free. We also conclude that there does not exist a controller which regularly implements $\mathcal{K}$ and in which (c_2, c_3) is free. As $\mathtt{p}((\mathcal{P}_{\text{full}} \wedge_w \mathcal{K})_c) = \operatorname{rank}(P(\xi)) - \operatorname{rank}\begin{pmatrix} 1 & \xi \\ 0 & 1 \\ 0 & 0 \\ 1 & \xi \end{pmatrix} = 2$ and $\mathtt{k}_1$(the cardinality of c_3) $= 1$, we have $\mathtt{p}((\mathcal{P}_{\text{full}} \wedge_w \mathcal{K})_c) = \mathtt{k}_1 + \mathtt{p}(\mathcal{P}_{\text{full}})_c$. Therefore from Theorem 5.10, $\mathcal{K}$ is indeed regularly implementable using a controller in which (c_1, c_2) is input (and c_3 is output). A controller which regularly implements $\mathcal{K}$ and in which (c_1, c_2) is input is found as follows. We have $(\mathcal{P}_{\text{full}})_c = \ker(P_c(d/dt))$ and $\mathcal{P}_{\text{full}} \wedge_w \mathcal{K} = \ker(P_k(d/dt))$ where $P_c(\xi) = (0 \quad \xi \quad 1)$ and $P_k(\xi) = \begin{pmatrix} 1 & \xi & 0 & 1 & \xi \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \xi & 1 \\ 1 & \xi & 0 & 0 & 0 \end{pmatrix}$ respectively. Eliminating w from $\mathcal{P}_{\text{full}} \wedge_w \mathcal{K}$ yields $(\mathcal{P}_{\text{full}} \wedge_w \mathcal{K})_c$ which is given by $(\mathcal{P}_{\text{full}} \wedge_w \mathcal{K})_c = \ker(P_{kc}(d/dt))$, where $P_{kc}(\xi) = \begin{pmatrix} 0 & \xi & 1 \\ 0 & 1 & \xi \end{pmatrix}$. By direct inspection we see that $\mathcal{C} := \ker(C(d/dt))$ where $C(\xi) = (0 \quad 1 \quad \xi)$ regularly implements $(\mathcal{P}_{\text{full}} \wedge_w \mathcal{K})_c$ w.r.t. $(\mathcal{P}_{\text{full}})_c$. The same C regularly implements $\mathcal{K}$ through (c_1, c_2, c_3) w.r.t. $\mathcal{P}_{\text{full}}$ and (c_1, c_2) is input in C.

2) Example 7.2: Let $\mathcal{P}_{\text{full}}$ with manifest variable $w = (w_1, w_2)$ and control variable $c = (c_1, c_2, c_3)$ be represented by the equations $w_1 + \dot{w}_2 + \dot{c}_3 = 0$, $w_2 + c_1 + c_2 + c_3 = 0$. Clearly $(\mathcal{P}_{\text{full}})_w = \mathfrak{C}(\mathbb{R}, \mathbb{R}^2)$ and $(\mathcal{P}_{\text{full}})_c = \mathfrak{C}(\mathbb{R}, \mathbb{R}^2)$. $(\mathcal{P}_{\text{full}})_w$ is trivially stabilizable, and w is detectable from c in $\mathcal{P}_{\text{full}}$. $\mathtt{p}((\mathcal{P}_{\text{full}})_c) = 0$. We compute $\mathcal{N}_c(\mathcal{P}_{\text{full}}) = \ker(N(d/dt))$, $\mathcal{N}_{(c_1,c_2)}(\mathcal{P}_{\text{full}}) = \ker(N_{12}(d/dt))$ and $\mathcal{N}_{(c_2,c_3)}(\mathcal{P}_{\text{full}}) = \ker(N_{23}(d/dt))$ where $N(\xi) = \begin{pmatrix} 0 & 0 & \xi \\ 1 & 1 & 1 \end{pmatrix}$, $N_{12}(\xi) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $N_{23}(\xi) = \begin{pmatrix} 0 & \xi \\ 1 & 1 \end{pmatrix}$. Then $\mathtt{p}(\mathcal{N}_c(\mathcal{P}_{\text{full}})) = \operatorname{rank}(N) = 2$, $\mathtt{p}(\mathcal{N}_{(c_2,c_3)}(\mathcal{P}_{\text{full}})) = \operatorname{rank}(N_{23}) = 2$, and $\mathtt{p}(\mathcal{N}_{(c_1,c_2)}(\mathcal{P}_{\text{full}})) = \operatorname{rank}(N_{12}) = 1$. From these calculations it is evident that $\mathtt{p}(\mathcal{N}_c(\mathcal{P}_{\text{full}})) - \mathtt{p}(\mathcal{N}_{(c_2,c_3)}(\mathcal{P}_{\text{full}})) = \mathtt{p}((\mathcal{P}_{\text{full}})_c)$ and $\mathtt{p}(\mathcal{N}_c(\mathcal{P}_{\text{full}})) - \mathtt{p}(\mathcal{N}_{(c_1,c_2)}(\mathcal{P}_{\text{full}})) > \mathtt{p}((\mathcal{P}_{\text{full}})_c)$. Therefore from Theorem 6.3 we conclude that the plant is stabilizable using a controller in which c_1 is free. We also conclude that there does not exist a controller which stabilizes the plant and in which c_3 is free. A stabilizing controller in which c_1 is free can be found in the same way as given in the previous example, for any given regularly implementable and stable $\mathcal{K}$.

VIII. Conclusion

In this paper we have studied the problems of regular implementability and stabilization using controllers in which a pre-specified subset of the plant control variables should be free in the controller, and the problems of regular implementability and stabilization using controllers with a pre-specified input/output structure. Necessary and sufficient conditions have been derived for these problems in both the full and the partial interconnection case.

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